
**stichting
mathematisch
centrum**



AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 134/79 DECEMBER

A.E. BROUWER & A.M. COHEN

THE POINCARÉ SERIES OF THE POLYNOMIALS
INVARIANT UNDER SU_2 IN ITS IRREDUCIBLE
REPRESENTATION OF DEGREE ≤ 17

2e boerhaavestraat 49 amsterdam

BIBLIOTHEEK MATHEMATISCH CENTRUM
—AMSTERDAM—



Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

The Poincaré series of the polynomials invariant under SU_2 in its irreducible representation of degree ≤ 17

by

Andries E. Brouwer & Arjeh M. Cohen

ABSTRACT

Recently T.A. Springer established a formula for the Poincaré series of the invariant polynomials of the compact Lie group SU_2 corresponding to its irreducible representation of given degree $d+1$. The formula expresses the Poincaré series as a rational function that is presented as a sum of several other straightforwardly computable rational functions. By use of a computer these functions have been determined for all $d \leq 16$.

KEY WORDS & PHRASES: Poincaré series, SU_2 , invariant theory

INTRODUCTION

Springer's formula (cf. [3]) for the Poincaré series $P_d(z)$ of the SU_2 -invariant polynomials associated with its $(d+1)$ -dimensional irreducible representation reads as follows.

$$P_d(z) = \sum_{0 \leq j < \frac{d}{2}} (-1)^j \phi_{d-2j} \left(\frac{(1-z^2)^j z^{j(j+1)}}{(j, z^2)! (d-j, z^2)!} \right),$$

where for $n \in \mathbb{N}$ the operator ϕ_n transforms a rational function f in z to the rational function $\phi_n f$ determined by

$$(\phi_n f)(z^n) = \frac{1}{n} \sum_{j=1}^n f(e^{2\pi i j/n} z),$$

and where $(n, z)!$ is the polynomial function given by $(n, z)! = (1-z)(1-z^2) \dots (1-z^n)$.

About the Calculation

Clearly, if $f = \sum_{j=0}^N f_j z^j$ is a polynomial function in z , then we have

$$(1) \quad (\phi_n f) = \sum_{j=0}^{\lfloor N/n \rfloor} f_{jn} z^j,$$

where $\lfloor N/n \rfloor$ is the greatest integer $\leq N/n$.

Suppose that fg is the product of two rational functions one of which, say g , is a function in z^n , in other words there exists a function h in z such that $h(z^n) = g(z)$. Then

$$(2) \quad \phi_n(fg) = h\phi_n(f)$$

As the denominator of the ϕ_{d-2j} -argument in a summand of $P_d(z)$ is a product of factors of the form $1 - z^m$, we may, after suitable multiplication of denominator and numerator by common factors, assume that the denominator is a product of factors having the form $1 - z^{(d-2j)m}$. Application of (2) reduces the summand in $P_d(z)$ to the quotient of the ϕ_{d-2j} -image of a polynomial by

a product of terms of the form $1-z^m$.

Using (1), the numerator is readily computed, whereafter we have obtained a summand that is expressed as a rational function with integral coefficients. Addition of these summands (while keeping track of the signs $(-1)^j$ that appear in the formula) establishes the results.

RESULTS. The $P_d(z)$ for $d \leq 16$ are listed in the table below in a so-called representative form (cf. [4]). The interested reader may find the individual summands in the appendix.

TABLE

The Poincaré series of the invariants under SU_2 in its irreducible representation of degree $d+1 \leq 17$.

$d = 2$	$\frac{1}{1-z^2}$
$d = 3$	$\frac{1}{1-z^4}$
$d = 4$	$\frac{1}{(1-z^2)^2 (1-z^3)}$
$d = 5$	$\frac{1+z^{18}}{(1-z^4)^2 (1-z^8) (1-z^{12})}$
$d = 6$	$\frac{1+z^{15}}{(1-z^2)^2 (1-z^4)^2 (1-z^6) (1-z^{10})}$
$d = 7$	$\frac{1+2z^8+4z^{12}+4z^{14}+5z^{16}+9z^{18}+6z^{20}+9z^{22}+8z^{24}+9z^{26}+6z^{28}+9z^{30}+5z^{32}+4z^{34}+4z^{36}+2z^{40}+z^{48}}{(1-z^4)^2 (1-z^8)^2 (1-z^{12})^2 (1-z^{20})}$
$d = 8$	$\frac{1+z^8+z^9+z^{10}+z^{18}}{(1-z^2)^2 (1-z^3)^2 (1-z^4)^2 (1-z^5) (1-z^6) (1-z^7)}$

$$\begin{aligned}
 d = 9 \quad & \frac{1+z^4+5z^8+4z^{10}+17z^{12}+20z^{14}+47z^{16}+61z^{18}+97z^{20}+120z^{22}+}{(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})} \\
 & 165z^{24}+189z^{26}+223z^{28}+241z^{30}+254z^{32}+254z^{34}+241z^{36}+ \\
 & 223z^{38}+189z^{40}+165z^{42}+120z^{44}+97z^{46}+61z^{48}+47z^{50}+20z^{52}+ \\
 & 17z^{54}+4z^{56}+5z^{58}+z^{62}+z^{66}
 \end{aligned}$$

$$\begin{aligned}
 d = 10 \quad & \frac{1+2z^6+4z^8+4z^9+7z^{10}+8z^{11}+15z^{12}+15z^{13}+20z^{14}+27z^{15}+29z^{16}+}{(1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^8)(1-z^9)(1-z^{10})(1-z^{14})} \\
 & 35z^{17}+40z^{18}+44z^{19}+47z^{20}+55z^{21}+52z^{22}+57z^{23}+56z^{24}+57z^{25}+ \\
 & 52z^{26}+55z^{27}+47z^{28}+44z^{29}+40z^{30}+35z^{31}+29z^{32}+27z^{33}+20z^{34}+ \\
 & 15z^{35}+15z^{36}+8z^{37}+7z^{38}+4z^{39}+4z^{40}+2z^{42}+z^{48}
 \end{aligned}$$

$$\begin{aligned}
 d = 11 \quad & \frac{1+z^4+10z^8+12z^{10}+57z^{12}+95z^{14}+233z^{16}+394z^{18}+745z^{20}+}{(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})} \\
 & 1173z^{22}+1925z^{24}+2814z^{26}+4139z^{28}+5669z^{30}+7664z^{32}+ \\
 & 9821z^{34}+12390z^{36}+14953z^{38}+17692z^{40}+20204z^{42}+22537z^{44}+ \\
 & 24378z^{46}+25746z^{48}+26413z^{50}+26413z^{52}+25746z^{54}+24378z^{56}+ \\
 & 22537z^{58}+20204z^{60}+17692z^{62}+14953z^{64}+12390z^{66}+9821z^{68}+ \\
 & 7664z^{70}+5669z^{72}+4139z^{74}+2814z^{76}+1925z^{78}+1173z^{80}+745z^{82}+ \\
 & 394z^{84}+233z^{86}+95z^{88}+57z^{90}+12z^{92}+10z^{94}+z^{98}+z^{102}
 \end{aligned}$$

$$\begin{aligned}
 d = 12 \quad & \frac{1+z^3+z^4+3z^6+5z^7+7z^8+11z^9+21z^{10}+25z^{11}+40z^{12}+55z^{13}+75z^{14}+}{(1-z^2)(1-z^4)(1-z^5)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)} \\
 & 96z^{15}+130z^{16}+156z^{17}+194z^{18}+228z^{19}+267z^{20}+294z^{21}+330z^{22}+ \\
 & 345z^{23}+364z^{24}+365z^{25}+364z^{26}+345z^{27}+330z^{28}+294z^{29}+ \\
 & 267z^{30}+228z^{31}+194z^{32}+156z^{33}+130z^{34}+96z^{35}+75z^{36}+55z^{37}+ \\
 & 40z^{38}+25z^{39}+21z^{40}+11z^{41}+7z^{42}+5z^{43}+3z^{44}+z^{46}+z^{47}+z^{50}
 \end{aligned}$$

$$\begin{aligned}
d = 13 \quad & 1+z^4+19z^8+32z^{10}+156z^{12}+340z^{14}+916z^{16}+1913z^{18}+4115z^{20}+ \\
& 7741z^{22}+14544z^{24}+25118z^{26}+42489z^{28}+68114z^{30}+106498z^{32}+ \\
& 159697z^{34}+233971z^{36}+331412z^{38}+458878z^{40}+617966z^{42}+ \\
& 814543z^{44}+1047676z^{46}+1321155z^{48}+1629760z^{50}+1972851z^{52}+ \\
& 2341055z^{54}+2728439z^{56}+3120893z^{58}+3509075z^{60}+3876056z^{62}+ \\
& 4210210z^{64}+4496206z^{66}+4723183z^{68}+4879836z^{70}+4960463z^{72}+ \\
& 4960463z^{74}+4879836z^{76}+4723183z^{78}+4496206z^{80}+4210210z^{82}+ \\
& 3876056z^{84}+3509075z^{86}+3120893z^{88}+2728439z^{90}+2341055z^{92}+ \\
& 1972851z^{94}+1629760z^{96}+1321155z^{98}+1047676z^{100}+814543z^{102}+ \\
& 617966z^{104}+458878z^{106}+331412z^{108}+233971z^{110}+159697z^{112}+ \\
& 106498z^{114}+68114z^{116}+42489z^{118}+25118z^{120}+14544z^{122}+ \\
& 7741z^{124}+4115z^{126}+1913z^{128}+916z^{130}+340z^{132}+156z^{134}+ \\
& 32z^{136}+19z^{138}+z^{142}+z^{146} \\
& \hline
& (1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18}) \\
& (1-z^{20})(1-z^{22})(1-z^{24})
\end{aligned}$$

$$\begin{aligned}
d = 14 \quad & 1+z^4+5z^6+3z^7+18z^8+22z^9+55z^{10}+77z^{11}+158z^{12}+227z^{13}+ \\
& 396z^{14}+579z^{15}+908z^{16}+1286z^{17}+1904z^{18}+2609z^{19}+3655z^{20}+ \\
& 4865z^{21}+6502z^{22}+8386z^{23}+10800z^{24}+13480z^{25}+16765z^{26}+ \\
& 20342z^{27}+24480z^{28}+28872z^{29}+33748z^{30}+38709z^{31}+43994z^{32}+ \\
& 49174z^{33}+54374z^{34}+59258z^{35}+63866z^{36}+67854z^{37}+71317z^{38}+ \\
& 73958z^{39}+75790z^{40}+76724z^{41}+76724z^{42}+75790z^{43}+73958z^{44}+ \\
& 71317z^{45}+67854z^{46}+63866z^{47}+59258z^{48}+54374z^{49}+49174z^{50}+ \\
& 43994z^{51}+38709z^{52}+33748z^{53}+28872z^{54}+24480z^{55}+20342z^{56}+ \\
& 16765z^{57}+13480z^{58}+10800z^{59}+8386z^{60}+6502z^{61}+4865z^{62}+ \\
& 3655z^{63}+2609z^{64}+1904z^{65}+1286z^{66}+908z^{67}+579z^{68}+396z^{69}+ \\
& 227z^{70}+158z^{71}+77z^{72}+55z^{73}+22z^{74}+18z^{75}+3z^{76}+5z^{77}+z^{79}+ \\
& z^{83} \\
& \hline
& (1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)(1-z^{10}) \\
& (1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})
\end{aligned}$$

$$\begin{aligned}
d = 15 \quad & 1+2z^4+z^6+32z^8+78z^{10}+378z^{12}+1027z^{14}+3124z^{16}+7672z^{18}+ \\
& 18676z^{20}+40784z^{22}+86197z^{24}+170002z^{26}+323704z^{28}+ \\
& 586665z^{30}+1029080z^{32}+1737391z^{34}+2849501z^{36}+4530000z^{38}+ \\
& 7019920z^{40}+10596388z^{42}+15636688z^{44}+22556344z^{46}+ \\
& 31885923z^{48}+44179876z^{50}+60104681z^{52}+80312694z^{54}+ \\
& 105538484z^{56}+136437227z^{58}+173688963z^{60}+217804048z^{62}+ \\
& 269239746z^{64}+328183158z^{66}+394684584z^{68}+468435861z^{70}+ \\
& 548927416z^{72}+635239454z^{74}+726230157z^{76}+820357306z^{78}+ \\
& 915894836z^{80}+1010799973z^{82}+1102948881z^{84}+1190050285z^{86}+ \\
& 1269884526z^{88}+1340258093z^{90}+1399218002z^{92}+1445037255z^{94}+ \\
& 1476376889z^{96}+1492284743z^{98}+1492284743z^{100}+ \\
& 1476376889z^{102}+1445037255z^{104}+1399218002z^{106}+ \\
& 1340258093z^{108}+1269884526z^{110}+1190050285z^{112}+ \\
& 1102948881z^{114}+1010799973z^{116}+915894836z^{118}+ \\
& 820357306z^{120}+726230157z^{122}+635239454z^{124}+548927416z^{126}+ \\
& 468435861z^{128}+394684584z^{130}+328183158z^{132}+269239746z^{134}+ \\
& 217804048z^{136}+173688963z^{138}+136437227z^{140}+105538484z^{142}+ \\
& 80312694z^{144}+60104681z^{146}+44179876z^{148}+31885923z^{150}+ \\
& 22556344z^{152}+15636688z^{154}+10596388z^{156}+7019920z^{158}+ \\
& 4530000z^{160}+2849501z^{162}+1737391z^{164}+1029080z^{166}+ \\
& 586665z^{168}+323704z^{170}+170002z^{172}+86197z^{174}+40784z^{176}+ \\
& 18676z^{178}+7672z^{180}+3124z^{182}+1027z^{184}+378z^{186}+78z^{188}+ \\
& 32z^{190}+z^{192}+2z^{194}+z^{198} \\
& \hline
& (1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18}) \\
& (1-z^{20})(1-z^{22})(1-z^{24})(1-z^{26})(1-z^{28})
\end{aligned}$$

$$\begin{aligned}
d = 16 \quad & 1+z^3+z^4+2z^5+8z^6+11z^7+30z^8+59z^9+112z^{10}+204z^{11}+389z^{12}+ \\
& 655z^{13}+1146z^{14}+1896z^{15}+3067z^{16}+4823z^{17}+7477z^{18}+11167z^{19}+ \\
& 16506z^{20}+23764z^{21}+33593z^{22}+46576z^{23}+63588z^{24}+85049z^{25}+ \\
& 112273z^{26}+145668z^{27}+186298z^{28}+234773z^{29}+292016z^{30}+ \\
& 357883z^{31}+433493z^{32}+518058z^{33}+611720z^{34}+713572z^{35}+ \\
& 822997z^{36}+937742z^{37}+1057289z^{38}+1178403z^{39}+1299472z^{40}+ \\
& 1417560z^{41}+1530508z^{42}+1634576z^{43}+1728646z^{44}+1808808z^{45}+ \\
& 1873856z^{46}+1921520z^{47}+1951002z^{48}+1960390z^{49}+1951002z^{50}+ \\
& 1921520z^{51}+1873856z^{52}+1808808z^{53}+1728646z^{54}+1634576z^{55}+ \\
& 1530508z^{56}+1417560z^{57}+1299472z^{58}+1178403z^{59}+1057289z^{60}+ \\
& 937742z^{61}+822997z^{62}+713572z^{63}+611720z^{64}+ \\
& 518058z^{65}+433493z^{66}+357883z^{67}+292016z^{68}+234773z^{69}+ \\
& 186298z^{70}+145668z^{71}+112273z^{72}+85049z^{73}+63588z^{74}+ \\
& 46576z^{75}+33593z^{76}+23764z^{77}+16506z^{78}+11167z^{79}+7477z^{80}+ \\
& 4823z^{81}+3067z^{82}+1896z^{83}+1146z^{84}+655z^{85}+389z^{86}+204z^{87}+ \\
& 112z^{88}+59z^{89}+30z^{90}+11z^{91}+8z^{92}+2z^{93}+z^{94}+z^{95}+z^{98} \\
& \hline
& (1-z^2)(1-z^4)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^7)(1-z^8)(1-z^9) \\
& (1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})(1-z^{15})
\end{aligned}$$

REMARKS.

- (i) The series $P_d(z)$ for $d \leq 10$ and $d = 12$ appear in [4]. They agree with the contents of the table. Results for $d \leq 12$ are also available in [1,2], but we have not compared them with ours. The expressions of $P_d(z)$ for $13 \leq d \leq 16$ seem to be new.
- (ii) The expression given for $P_d(z)$ in the appendix has the property that the numerator and denominator are not simultaneously divisible by a polynomial of the form $(1-z^m)/(1-z^\ell)$ (ℓ divisor of m) for m such that $1-z^m$ occurs as an explicitly given factor in the denominator. The representative form however is easily obtained from the given one.

REFERENCES

- [1] JARUŠEK, J., A generating function for the number of invariants of a binary form of the 11th order in Petr's normal form, Rozprawy II, Tridy České. Akad. 57, no. 8, pp. 17, (1947).

- [2] PETR, K., *On generating functions for the number of invariants belonging to one fundamental binary form*, Rozpravy II, Trídy Česk^V Akad. 56, no. 5, pp. 23, (1946).

Generalization of results on the generating function giving the number of invariants for a binary form of a given degree, Rozpravy II, Trídy Česk^V Akad. 56, no. 8, pp. 22, (1946).

On the generating function in normal form for the number of invariants of a binary form of degree 12, Rozpravy II, Trídy Česk^V Akad. 56, no. 10, pp. 16, (1946).

On the generating functions for the number of invariants belonging to a system of two binary forms, Rozpravy II, Trídy Česk^V Akad. 56, no. 14, pp. 15, (1946).

- [3] SPRINGER, T.A., *On the invariant theory of Sl_2* , preprint.

- [4] SYLVESTER, J.J., *The Collected Mathematical Papers of J.J. Sylvester*, Cambridge Univ. Press 1909, vol. III, pp. 283-311, 489-508.

D=2; J=0
 NUMERATOR:
 +1Z0
 DENOMINATOR:
 (1-Z2)

*****THE POINCARÉ SERIES FOR D=2*****:

NUMERATOR:
 +1Z0
 DENOMINATOR:
 (1-Z2)

D=3; J=0
 NUMERATOR:
 +1Z0
 DENOMINATOR:
 (1-Z4)(1-Z2)

D=3; J=1
 NUMERATOR:
 +1Z2
 DENOMINATOR:
 (1-Z2)(1-Z4)

*****THE POINCARÉ SERIES FOR D=3*****:

NUMERATOR:
 +1Z0
 DENOMINATOR:
 (1-Z4)

D=4; J=0
 NUMERATOR:
 +1Z0
 DENOMINATOR:
 (1-Z1)(1-Z3)(1-Z2)

D=4; J=1
 NUMERATOR:
 +1Z1
 DENOMINATOR:
 (1-Z1)(1-Z2)(1-Z3)

*****THE POINCARÉ SERIES FOR D=4*****:

NUMERATOR:
 +1Z0
 DENOMINATOR:
 (1-Z2)(1-Z3)

D=5; J=0
 NUMERATOR:
 +1Z0+1Z2+4Z4+5Z6+7Z8+4Z10+3Z12
 DENOMINATOR:
 (1-Z4)(1-Z6)(1-Z8)(1-Z2)

```

D=5;      J=1
NUMERATOR:
+2Z2+2Z4+3Z6+1Z8+1Z10
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z8)

D=5;      J=2
NUMERATOR:
+1Z5
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z6)

*****THE POINCARÉ SERIES FOR D=5*****
NUMERATOR:
+1Z0-1Z6+1Z12
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)

D=6;      J=0
NUMERATOR:
+1Z0+1Z2+2Z3+2Z4+1Z5+2Z6
DENOMINATOR:
(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)

D=6;      J=1
NUMERATOR:
+1Z1+1Z2+1Z3+1Z5
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)

D=6;      J=2
NUMERATOR:
+1Z3
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z2)(1-Z3)(1-Z4)

*****THE POINCARÉ SERIES FOR D=6*****
NUMERATOR:
+1Z0-1Z2-1Z3+1Z5+1Z7-1Z9
DENOMINATOR:
(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z5)

D=7;      J=0
NUMERATOR:
+1Z0+3Z2+19Z4+4Z6+9Z8+156Z10+236Z12+288Z14+335Z16+329Z18+304Z20+235Z22+174Z24+100Z26+54Z28+20Z30+6Z32
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)

D=7;      J=1
NUMERATOR:
+5Z2+16Z4+38Z6+60Z8+88Z10+93Z12+103Z14+85Z16+69Z18+41Z20+21Z22+5Z24+1Z26
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)

D=7;      J=2
NUMERATOR:

```

+172+224+926+1178+15710+15712+14214+7216+6718+1720
 DENOMINATOR:
 (1-72)(1-74)(1-74)(1-72)(1-78)(1-710)

D=7; J=3
 NUMERATOR:
 +1712
 DENOMINATOR:
 (1-72)(1-74)(1-75)(1-74)(1-76)(1-78)

*****THE POINCARÉ SERIES FOR D=7*****
 NUMERATOR:
 +170-176+278-1710+5712+2714+6215+2718+5720-1722+2724-1726+1732
 DENOMINATOR:
 (1-74)(1-75)(1-78)(1-710)(1-712)

D=8; J=0
 NUMERATOR:
 +170+372+473+774+775+976+877+978+679+5710+3711+2712
 DENOMINATOR:
 (1-71)(1-73)(1-71)(1-75)(1-73)(1-77)(1-72)

D=8; J=1
 NUMERATOR:
 +271+372+573+874+1075+1076+1277+1078+879+7710+4711+1712+1713
 DENOMINATOR:
 (1-71)(1-72)(1-71)(1-74)(1-75)(1-72)(1-77)

D=8; J=2
 NUMERATOR:
 +172+173+174+176
 DENOMINATOR:
 (1-71)(1-71)(1-71)(1-73)(1-72)(1-75)(1-73)

D=6; J=3
 NUMERATOR:
 +176
 DENOMINATOR:
 (1-71)(1-72)(1-73)(1-72)(1-73)(1-74)(1-75)

*****THE POINCARÉ SERIES FOR D=8*****
 NUMERATOR:
 +170-172-173+175+176-1711-1712+1714+1715-1717
 DENOMINATOR:
 (1-72)(1-72)(1-73)(1-73)(1-74)(1-75)(1-77)

D=0; J=0
 NUMERATOR:
 +170+672+4924+16076+1478+818210+1451712+2223714+3164716+4099218+5021720+568722+6105724+6092726+5773728+5077730+4235732+3244734+2341736
 +1513738+893740+438242+188744+53746+1748
 DENOMINATOR:
 (1-74)(1-72)(1-78)(1-710)(1-74)(1-714)(1-716)(1-72)

D=9; J=1
 NUMERATOR:
 +1172+6774+25676+67078+1452710+2629712+4268714+6160716+8247718+1010720+11636722+12411724+12531726+11742728+10395730+8468732+6490734+449


```

DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)

*****THE POINCARÉ SERIES FOR D=10*****
NUMERATOR:
+120-172-123+376+278+279+4710+5712-1713+1714+3716-4717+4718-3719-1721+1722-6723-4725-2726-2727-3729+1732+1733-1735
DENOMINATOR:
(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)

D=11; J=0
NUMERATOR:
+170+1312+153274+85176+351678+11108710+2973712+69185714+145026716+277592718+493866720+822988722+1297562724+1944356726+278606828+3828074
730+503386732+6458374734+7964750736+9506816738+11002659740+1235360742+13472597744+14273622746+14702066748+14719445750+1433007752+1355
8279254+12467535756+11131643758+9647262760+8103600762+6592623764+518338766+3933427768+277189770+2013147772+1348408774+860020776+518372
778+293571780+15421782+74416784+32197786+12274288+3912790+1004792+177794+20796
DENOMINATOR:
(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)

D=11; J=1
NUMERATOR:
+2272+201274+98276+316478+8046710+17042712+31863714+53525716+83060218+119965720+153532222+211257724+260380726+306277728+345814730+3743397
32+390158734+390706736+37030738+34933740+311295742+265496744+216902746+168610748+124653750+86697752+56499754+33846156+18424758+8780760
+3966762+1127764+251766+27768+1270
DENOMINATOR:
(1-22)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)

D=11; J=2
NUMERATOR:
+822+9424+56976+210278+6071710+14204712+28992714+52249716+86119718+130619720+185535722+247579724+313406726+376697728+4432357730+473515732
+496598734+497895236+478268238+438986740+385509742+322512744+257075746+193851748+138040750+91681752+56530754+31633756+15913758+6887760+2
508262+685264+133766+11768
DENOMINATOR:
(1-22)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)

D=11; J=3
NUMERATOR:
+974+5176+26278+770210+1863712+3709714+6596716+10459218+15366720+20812722+26538724+31595726+35635728+37787730+38081732+35148734+32597736
+27549738+22018740+16373742+11408744+7254746+4228748+2143250+946752+329754+87756+12758+1760
DENOMINATOR:
(1-22)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)

D=11; J=4
NUMERATOR:
+328+5210+19712+29714+50716+6718+84720+99722+97724+84726+74728+54730+36732+20734+12736+2738+1740
DENOMINATOR:
(1-22)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)

D=11; J=5
NUMERATOR:
+1730
DENOMINATOR:
(1-22)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)

*****THE POINCARÉ SERIES FOR D=11*****
NUMERATOR:
+120+174-176+1726+112710+58712+85714+222716+336718+660720+951222+1589724+2154726+3188728+4080730+5510732+6633734+8310736+9443738+11059740
+11894742+13094744+13319746+13852748+13319750+13094752+11894754+11059756+9443758+8310760+6633762+5510764+4080766+3188768+2154770+1589772
+951774+660776+336778+222780+85782+58784+11786+10788-1790+1792+1796

```

DENOMINATOR:
(1-24)(1-26)(1-78)(1-210)(1-212)(1-214)(1-218)(1-220)

D=12; J=0
NUMERATOR:
+120+121+1072+2673+5524+10675+18176+27677+39328+52079+64710+76421+118652+129332+13+97271+943716+876217+784718+667219+541220+41322
+1296222+194723+11724+63725+29726+11227+4728
DENOMINATOR:
(1-21)(1-21)(1-22)(1-25)(1-71)(1-77)(1-74)(1-73)(1-75)(1-71)(1-72)

D=12; J=1
NUMERATOR:
+51+2022+6673+16874+37225+71726+12797+210678+326879+4792710+6716711+8991212+11595113+1439714+17288715+20680216+22533217+24729718+2626
871+27094220+27179721+28482722+25099723+23082724+20624725+17854726+14979727+12135728+9498229+7136730+5149231+3540732+2314733+141773+81
4735+424236+201737+81738+27239+5240+1241
DENOMINATOR:
(1-21)(1-22)(1-23)(1-74)(1-71)(1-76)(1-77)(1-78)(1-79)(1-22)(1-21)

D=12; J=2
NUMERATOR:
+121+722+1723+3724+5675+11276+16177+22128+27879+332710+369711+396712+394713+382714+340715+293716+234717+179218+120219+80220+43221+22222+
7723+3724
DENOMINATOR:
(1-21)(1-21)(1-21)(1-23)(1-21)(1-25)(1-23)(1-27)(1-22)(1-29)(1-75)

D=12; J=3
NUMERATOR:
+172+223+974+1575+2676+3577+5278+6079+74710+77211+80212+72213+69214+52215+42216+29217+19218+8219+6720+1721
DENOMINATOR:
(1-21)(1-22)(1-21)(1-22)(1-21)(1-24)(1-25)(1-22)(1-77)(1-78)(1-73)

D=12; J=4
NUMERATOR:
+175+277+278+379+2710+2711+1712+2713+1714
DENOMINATOR:
(1-21)(1-21)(1-23)(1-22)(1-21)(1-23)(1-22)(1-25)(1-23)(1-27)(1-24)

D=12; J=5
NUMERATOR:
+1715
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-72)(1-23)(1-24)(1-25)(1-76)(1-77)

*****THE POINCARÉ SERIES FOR D=12*****

NUMERATOR:
+120-172+174+276+477+478+479+10710+10711+1512+20713+25714+26715+35716+35717+38718+37719+38720+28721+26722+13723+6724-6725-13726-26727-2
8728-38729-37230-38231-35732-35733-26734-25735-20736-15737-10738-10739-4740-4741-4742-2743-1745+1747-1749
DENOMINATOR:
(1-22)(1-72)(1-23)(1-24)(1-25)(1-75)(1-26)(1-77)(1-78)(1-29)(1-71)

D=13; J=0
NUMERATOR:
+120+2322+37674+287076+1512378+60700710+201500212+574964714+1458029216+3350844218+7098549220+14019844722+26071711724+45976366726+7736784
5728+12492193730+193880853732+290870261734+42270430736+596397419738+818633011740+1694992432742+1429391191744+1823198363746+22767891027
48+277881211750+3326200519252+39038707227254+4495366115256+5081193883259+5640128954760+6150055700752+6589717995764+89939720382766+7184275
874768+7311905549770+7316702593772+7198328171774+5962478228276+662002460378+6186658269280+5681349032282+5125446905784+4540871714786+394

9131842788+3369733043790+2319239516792+2311771083794+18599430977296+457563177298+1119631012100+8378933447102+6116977292104+434483358710
 642996901817108+20025981971110+1292942397112+803799367114+479332202116+272816922118+14736482120+74966772122+35604037124+15990057126+6203
 502128+2192977136+670907132+169157134+33237136+4402138+337140
 DENOMINATOR:
 (1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)(1-720)(1-722)(1-724)(1-726)

D=13; J=1
 NUMERATOR:
 +4222+641744+84046+2404078+01109710+282918712+756462114+1794004716+3861266718+7659540720+14160466722+24722428724+408977197226+64559801728
 +9771860730+142356964732+200254644734+272702924736+3603245292736+46276928874+57854002742+70529596874+836996271746+974894735748+110744
 9465750+12306090752+1336384239254+1425189900756+1486402715259+1518598931760+1520028679762+149053535764+1431884069766+1347236811268+12
 41135432770+111911313072+98718916374+851328821776+717234070278+589748950280+472786093782+369031893784+280044645786+206220207768+147052
 452790+101270775792+67152619734+42708038796+25935191798+149495917100+81232622102+41218677104+19310207106+8217607108+31126667110+101665711
 2+274377114+56677115+7932119+547120+17122
 DENOMINATOR:
 (1-22)(1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)(1-720)(1-722)(1-724)(1-726)

D=13; J=2
 NUMERATOR:
 +1922+32324+245076+1108078+37352710+100760712+232740714+474244716+877302716+1496115220+2387579722+3595701724+5153833726+70642977228+93056
 79230+11912569732+14493758734+17216780736+19838851738+22195506740+2413961724+25530589744+2627595746+26313847248+25648539750+2432196875
 2+22436202754+20115136756+17517472758+14795406760+12105298752+9571389764+7298860766+5347939268+3752016770+2504759772+1581316774+93392577
 6+510391778+253052180+111544782+41978734+12923286+2917788+431790+26792
 DENOMINATOR:
 (1-22)(1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)(1-720)(1-722)

D=13; J=3
 NUMERATOR:
 +122+5324+449976+278828+10959710+34202712+89031714+202328716+4410514718+760221720+1301267722+2085019724+3150499726+4524096728+6200838730+8
 15008732+12077221734+12542590736+14748164238+1677189740+18459458742+19686236744+20346716745+203933381748+19815348750+18667267752+170353
 13254+15033430756+12862036758+10616110760+844599762+644802764+444696765+3329278768+2218624770+1397724772+824281774+450860776+22512977
 8+100988780+39527782+13098284+3436786+66178+74290+3792
 DENOMINATOR:
 (1-22)(1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)(1-720)

D=13; J=4
 NUMERATOR:
 +124+1376+11326+500710+1726712+4698714+11102716+27854718+42814726+73244722+116948724+174808726+247443228+332087730+425609732+521008734+6
 12010736+689659238+747365740+77957742+782912744+756152746+703365748+628585750+540083252+444497754+350481756+263309758+188354760+1272257
 62+80987264+47864766+26192758+12890770+5680772+2119774+665776+150779+25780+1782
 DENOMINATOR:
 (1-22)(1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)

D=13; J=5
 NUMERATOR:
 +1716+2712+13714+28716+59718+101720+174722+245724+347726+438728+591732+646734+638736+620738+552740+473742+369744+288746+191748+12
 4250+71752+38754+13756+7758+1760
 DENOMINATOR:
 (1-22)(1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)

D=13; J=6
 NUMERATOR:
 +1242

DENOMINATOR:
 (1-22)(1-24)(1-26)(1-28)(1-710)(1-712)(1-714)(1-716)(1-718)

*****THE PUNICARE SERIES FOR D=13*****

NUMERATOR:
 +170-122+24-26+2178+11710+145712+195714+721716+1192718+2923720+4618722+9726724+15392726+27097728+41017730+65481732+94216734+139755736+
 191657238+267221740+35074742+63798744+583878746+737277748+892483750+1080368752+1260687254+1467752756+1653141758+1855934760+2020122762+

2190088764+2306118766+2617065768+2462771270+2497692272+2442771174+2417065776+2306118778+2190088780+2020122232+1855934284+1653141736+1467
 752288+126068790+1080368792+892483294+73727796+583878798+4637982100+3507452102+25722212104+1916577106+139752108+942162110+654817112+41
 0172114+27097116+153922118+97262120+46182122+29232124+11922126+7212128+1957130+1452132+117134+217136+27138+27140+17142+17144
 DENOMINATOR:
 (1-72)(1-78)(1-710)(1-712)(1-714)(1-716)(1-718)(1-720)(1-722)(1-724)

D=14; J=0
 NUMERATOR:
 +120+321+2822+11323+39074+107825+271776+604777+1257028+2445029+43995710+75811211+125325212+198553213+364184714+450521215+648585216+90793
 8217+12400347118+1653195219+2156821220+754202221+344697822+4236090223+5110360724+6055854225+7057225226+8087789727+9123203228+1012901072
 9+11076417230+11928639231+12658649732+13235643733+13639593734+13851565239+13867020736+13680951237+13305035238+12750170239+12042142740+11
 20371741+10259685742+9258538743+823696244+7202463245+6196750246+5240713247+4355569248+355293749+2848037250+2235868751+1719873752+1293
 490253+930952754+661331755+475649756+322151757+2116522558+134005159+81788760+47603761+26467762+13806763+6793764+3034765+1258266+4444767+14
 1768+33259+7770
 DENOMINATOR:
 (1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

D=14; J=1
 NUMERATOR:
 +721+2822+9423+24374+52925+102076+178277+286528+430429+6098210+8223711+10621212+13220213+15916214+18607215+21163716+23453717+25339718+26
 703219+27438220+27497221+26869222+25608223+23798224+21567225+19945226+16370227+13663728+11044729+860230+644731+4578732+3081733+1937734
 +1128235+595236+278737+112736+37239+7460+1761
 DENOMINATOR:
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

D=14; J=2
 NUMERATOR:
 +321+2022+10223+31274+86525+190126+392677+731028+1283679+21045210+32964211+49133212+70572213+97463214+130525215+169144716+213385217+2616
 6218+313191219+36939220+41693721+46460172+50692723+54092624+58537825+57824026+57972327+56868228+56675529+514225730+47379231+
 426634732+437634733+523460734+271919235+22260736+177718237+137675238+103694239+75410240+53054741+35765742+23168743+14186744+8261745+444
 7246+2236747+989748+397749+122750+33751+4252
 DENOMINATOR:
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

D=14; J=3
 NUMERATOR:
 +322+1223+3774+9625+19476+35677+59328+91729+1314210+1903211+2332212+2907713+34666714+3997215+44423715+4764717+4931718+4959219+4809220+4534
 721+4102222+3612223+3042224+2480225+1924726+1438727+1001228+670729+407730+236731+111732+52733+15234+5735
 DENOMINATOR:
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

D=14; J=4
 NUMERATOR:
 +324+625+2026+3227+5928+9829+161210+204211+292212+340213+432214+460215+535216+529217+564218+514219+507220+425221+387222+293223+246224+16
 5225+125226+70227+48728+21229+12230+2231+1232
 DENOMINATOR:
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

D=14; J=5
 NUMERATOR:
 +129+279+2710+2711+5712+5713+7214+7215+7216+7217+6718+4219+5220+3221+2722+1724
 DENOMINATOR:
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

D=14; J=6
 NUMERATOR:
 +1221
 DENOMINATOR:

(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)(1-86)(1-87)(1-88)(1-89)(1-90)(1-91)(1-92)(1-93)(1-94)(1-95)(1-96)(1-97)(1-98)(1-99)(1-100)

*****THE POINCARÉ SERIES FOR D=14*****

NUMERATOR:
 170+174+178+182+186+190+194+198+202+206+210+214+218+222+226+230+234+238+242+246+250+254+258+262+266+270+274+278+282+286+290+294+298+302+306+310+314+318+322+326+330+334+338+342+346+350+354+358+362+366+370+374+378+382+386+390+394+398+402+406+410+414+418+422+426+430+434+438+442+446+450+454+458+462+466+470+474+478+482+486+490+494+498+502+506+510+514+518+522+526+530+534+538+542+546+550+554+558+562+566+570+574+578+582+586+590+594+598+602+606+610+614+618+622+626+630+634+638+642+646+650+654+658+662+666+670+674+678+682+686+690+694+698+702+706+710+714+718+722+726+730+734+738+742+746+750+754+758+762+766+770+774+778+782+786+790+794+798+802+806+810+814+818+822+826+830+834+838+842+846+850+854+858+862+866+870+874+878+882+886+890+894+898+902+906+910+914+918+922+926+930+934+938+942+946+950+954+958+962+966+970+974+978+982+986+990+994+998+1002+1006+1010+1014+1018+1022+1026+1030+1034+1038+1042+1046+1050+1054+1058+1062+1066+1070+1074+1078+1082+1086+1090+1094+1098+1102+1106+1110+1114+1118+1122+1126+1130+1134+1138+1142+1146+1150+1154+1158+1162+1166+1170+1174+1178+1182+1186+1190+1194+1198+1202+1206+1210+1214+1218+1222+1226+1230+1234+1238+1242+1246+1250+1254+1258+1262+1266+1270+1274+1278+1282+1286+1290+1294+1298+1302+1306+1310+1314+1318+1322+1326+1330+1334+1338+1342+1346+1350+1354+1358+1362+1366+1370+1374+1378+1382+1386+1390+1394+1398+1402+1406+1410+1414+1418+1422+1426+1430+1434+1438+1442+1446+1450+1454+1458+1462+1466+1470+1474+1478+1482+1486+1490+1494+1498+1502+1506+1510+1514+1518+1522+1526+1530+1534+1538+1542+1546+1550+1554+1558+1562+1566+1570+1574+1578+1582+1586+1590+1594+1598+1602+1606+1610+1614+1618+1622+1626+1630+1634+1638+1642+1646+1650+1654+1658+1662+1666+1670+1674+1678+1682+1686+1690+1694+1698+1702+1706+1710+1714+1718+1722+1726+1730+1734+1738+1742+1746+1750+1754+1758+1762+1766+1770+1774+1778+1782+1786+1790+1794+1798+1802+1806+1810+1814+1818+1822+1826+1830+1834+1838+1842+1846+1850+1854+1858+1862+1866+1870+1874+1878+1882+1886+1890+1894+1898+1902+1906+1910+1914+1918+1922+1926+1930+1934+1938+1942+1946+1950+1954+1958+1962+1966+1970+1974+1978+1982+1986+1990+1994+1998+2002+2006+2010+2014+2018+2022+2026+2030+2034+2038+2042+2046+2050+2054+2058+2062+2066+2070+2074+2078+2082+2086+2090+2094+2098+2102+2106+2110+2114+2118+2122+2126+2130+2134+2138+2142+2146+2150+2154+2158+2162+2166+2170+2174+2178+2182+2186+2190+2194+2198+2202+2206+2210+2214+2218+2222+2226+2230+2234+2238+2242+2246+2250+2254+2258+2262+2266+2270+2274+2278+2282+2286+2290+2294+2298+2302+2306+2310+2314+2318+2322+2326+2330+2334+2338+2342+2346+2350+2354+2358+2362+2366+2370+2374+2378+2382+2386+2390+2394+2398+2402+2406+2410+2414+2418+2422+2426+2430+2434+2438+2442+2446+2450+2454+2458+2462+2466+2470+2474+2478+2482+2486+2490+2494+2498+2502+2506+2510+2514+2518+2522+2526+2530+2534+2538+2542+2546+2550+2554+2558+2562+2566+2570+2574+2578+2582+2586+2590+2594+2598+2602+2606+2610+2614+2618+2622+2626+2630+2634+2638+2642+2646+2650+2654+2658+2662+2666+2670+2674+2678+2682+2686+2690+2694+2698+2702+2706+2710+2714+2718+2722+2726+2730+2734+2738+2742+2746+2750+2754+2758+2762+2766+2770+2774+2778+2782+2786+2790+2794+2798+2802+2806+2810+2814+2818+2822+2826+2830+2834+2838+2842+2846+2850+2854+2858+2862+2866+2870+2874+2878+2882+2886+2890+2894+2898+2902+2906+2910+2914+2918+2922+2926+2930+2934+2938+2942+2946+2950+2954+2958+2962+2966+2970+2974+2978+2982+2986+2990+2994+2998+3002+3006+3010+3014+3018+3022+3026+3030+3034+3038+3042+3046+3050+3054+3058+3062+3066+3070+3074+3078+3082+3086+3090+3094+3098+3102+3106+3110+3114+3118+3122+3126+3130+3134+3138+3142+3146+3150+3154+3158+3162+3166+3170+3174+3178+3182+3186+3190+3194+3198+3202+3206+3210+3214+3218+3222+3226+3230+3234+3238+3242+3246+3250+3254+3258+3262+3266+3270+3274+3278+3282+3286+3290+3294+3298+3302+3306+3310+3314+3318+3322+3326+3330+3334+3338+3342+3346+3350+3354+3358+3362+3366+3370+3374+3378+3382+3386+3390+3394+3398+3402+3406+3410+3414+3418+3422+3426+3430+3434+3438+3442+3446+3450+3454+3458+3462+3466+3470+3474+3478+3482+3486+3490+3494+3498+3502+3506+3510+3514+3518+3522+3526+3530+3534+3538+3542+3546+3550+3554+3558+3562+3566+3570+3574+3578+3582+3586+3590+3594+3598+3602+3606+3610+3614+3618+3622+3626+3630+3634+3638+3642+3646+3650+3654+3658+3662+3666+3670+3674+3678+3682+3686+3690+3694+3698+3702+3706+3710+3714+3718+3722+3726+3730+3734+3738+3742+3746+3750+3754+3758+3762+3766+3770+3774+3778+3782+3786+3790+3794+3798+3802+3806+3810+3814+3818+3822+3826+3830+3834+3838+3842+3846+3850+3854+3858+3862+3866+3870+3874+3878+3882+3886+3890+3894+3898+3902+3906+3910+3914+3918+3922+3926+3930+3934+3938+3942+3946+3950+3954+3958+3962+3966+3970+3974+3978+3982+3986+3990+3994+3998+4002+4006+4010+4014+4018+4022+4026+4030+4034+4038+4042+4046+4050+4054+4058+4062+4066+4070+4074+4078+4082+4086+4090+4094+4098+4102+4106+4110+4114+4118+4122+4126+4130+4134+4138+4142+4146+4150+4154+4158+4162+4166+4170+4174+4178+4182+4186+4190+4194+4198+4202+4206+4210+4214+4218+4222+4226+4230+4234+4238+4242+4246+4250+4254+4258+4262+4266+4270+4274+4278+4282+4286+4290+4294+4298+4302+4306+4310+4314+4318+4322+4326+4330+4334+4338+4342+4346+4350+4354+4358+4362+4366+4370+4374+4378+4382+4386+4390+4394+4398+4402+4406+4410+4414+4418+4422+4426+4430+4434+4438+4442+4446+4450+4454+4458+4462+4466+4470+4474+4478+4482+4486+4490+4494+4498+4502+4506+4510+4514+4518+4522+4526+4530+4534+4538+4542+4546+4550+4554+4558+4562+4566+4570+4574+4578+4582+4586+4590+4594+4598+4602+4606+4610+4614+4618+4622+4626+4630+4634+4638+4642+4646+4650+4654+4658+4662+4666+4670+4674+4678+4682+4686+4690+4694+4698+4702+4706+4710+4714+4718+4722+4726+4730+4734+4738+4742+4746+4750+4754+4758+4762+4766+4770+4774+4778+4782+4786+4790+4794+4798+4802+4806+4810+4814+4818+4822+4826+4830+4834+4838+4842+4846+4850+4854+4858+4862+4866+4870+4874+4878+4882+4886+4890+4894+4898+4902+4906+4910+4914+4918+4922+4926+4930+4934+4938+4942+4946+4950+4954+4958+4962+4966+4970+4974+4978+4982+4986+4990+4994+4998+5002+5006+5010+5014+5018+5022+5026+5030+5034+5038+5042+5046+5050+5054+5058+5062+5066+5070+5074+5078+5082+5086+5090+5094+5098+5102+5106+5110+5114+5118+5122+5126+5130+5134+5138+5142+5146+5150+5154+5158+5162+5166+5170+5174+5178+5182+5186+5190+5194+5198+5202+5206+5210+5214+5218+5222+5226+5230+5234+5238+5242+5246+5250+5254+5258+5262+5266+5270+5274+5278+5282+5286+5290+5294+5298+5302+5306+5310+5314+5318+5322+5326+5330+5334+5338+5342+5346+5350+5354+5358+5362+5366+5370+5374+5378+5382+5386+5390+5394+5398+5402+5406+5410+5414+5418+5422+5426+5430+5434+5438+5442+5446+5450+5454+5458+5462+5466+5470+5474+5478+5482+5486+5490+5494+5498+5502+5506+5510+5514+5518+5522+5526+5530+5534+5538+5542+5546+5550+5554+5558+5562+5566+5570+5574+5578+5582+5586+5590+5594+5598+5602+5606+5610+5614+5618+5622+5626+5630+5634+5638+5642+5646+5650+5654+5658+5662+5666+5670+5674+5678+5682+5686+5690+5694+5698+5702+5706+5710+5714+5718+5722+5726+5730+5734+5738+5742+5746+5750+5754+5758+5762+5766+5770+5774+5778+5782+5786+5790+5794+5798+5802+5806+5810+5814+5818+5822+5826+5830+5834+5838+5842+5846+5850+5854+5858+5862+5866+5870+5874+5878+5882+5886+5890+5894+5898+5902+5906+5910+5914+5918+5922+5926+5930+5934+5938+5942+5946+5950+5954+5958+5962+5966+5970+5974+5978+5982+5986+5990+5994+5998+6002+6006+6010+6014+6018+6022+6026+6030+6034+6038+6042+6046+6050+6054+6058+6062+6066+6070+6074+6078+6082+6086+6090+6094+6098+6102+6106+6110+6114+6118+6122+6126+6130+6134+6138+6142+6146+6150+6154+6158+6162+6166+6170+6174+6178+6182+6186+6190+6194+6198+6202+6206+6210+6214+6218+6222+6226+6230+6234+6238+6242+6246+6250+6254+6258+6262+6266+6270+6274+6278+6282+6286+6290+6294+6298+6302+6306+6310+6314+6318+6322+6326+6330+6334+6338+6342+6346+6350+6354+6358+6362+6366+6370+6374+6378+6382+6386+6390+6394+6398+6402+6406+6410+6414+6418+6422+6426+6430+6434+6438+6442+6446+6450+6454+6458+6462+6466+6470+6474+6478+6482+6486+6490+6494+6498+6502+6506+6510+6514+6518+6522+6526+6530+6534+6538+6542+6546+6550+6554+6558+6562+6566+6570+6574+6578+6582+6586+6590+6594+6598+6602+6606+6610+6614+6618+6622+6626+6630+6634+6638+6642+6646+6650+6654+6658+6662+6666+6670+6674+6678+6682+6686+6690+6694+6698+6702+6706+6710+6714+6718+6722+6726+6730+6734+6738+6742+6746+6750+6754+6758+6762+6766+6770+6774+6778+6782+6786+6790+6794+6798+6802+6806+6810+6814+6818+6822+6826+6830+6834+6838+6842+6846+6850+6854+6858+6862+6866+6870+6874+6878+6882+6886+6890+6894+6898+6902+6906+6910+6914+6918+6922+6926+6930+6934+6938+6942+6946+6950+6954+6958+6962+6966+6970+6974+6978+6982+6986+6990+6994+6998+7002+7006+7010+7014+7018+7022+7026+7030+7034+7038+7042+7046+7050+7054+7058+7062+7066+7070+7074+7078+7082+7086+7090+7094+7098+7102+7106+7110+7114+7118+7122+7126+7130+7134+7138+7142+7146+7150+7154+7158+7162+7166+7170+7174+7178+7182+7186+7190+7194+7198+7202+7206+7210+7214+7218+7222+7226+7230+7234+7238+7242+7246+7250+7254+7258+7262+7266+7270+7274+7278+7282+7286+7290+7294+7298+7302+7306+7310+7314+7318+7322+7326+7330+7334+7338+7342+7346+7350+7354+7358+7362+7366+7370+7374+7378+7382+7386+7390+7394+7398+7402+7406+7410+7414+7418+7422+7426+7430+7434+7438+7442+7446+7450+7454+7458+7462+7466+7470+7474+7478+7482+7486+7490+7494+7498+7502+7506+7510+7514+7518+7522+7526+7530+7534+7538+7542+7546+7550+7554+7558+7562+7566+7570+7574+7578+7582+7586+7590+7594+7598+7602+7606+7610+7614+7618+7622+7626+7630+7634+7638+7642+7646+7650+7654+7658+7662+7666+7670+7674+7678+7682+7686+7690+7694+7698+7702+7706+7710+7714+7718+7722+7726+7730+7734+7738+7742+7746+7750+7754+7758+7762+7766+7770+7774+7778+7782+7786+7790+7794+7798+7802+7806+7810+7814+7818+7822+7826+7830+7834+7838+7842+7846+7850+7854+7858+7862+7866+7870+7874+7878+7882+7886+7890+7894+7898+7902+7906+7910+7914+7918+7922+7926+7930+7934+7938+7942+7946+7950+7954+7958+7962+7966+7970+7974+7978+7982+7986+7990+7994+7998+8002+8006+8010+8014+8018+8022+8026+8030+8034+8038+8042+8046+8050+8054+8058+8062+8066+8070+8074+8078+8082+8086+8090+8094+8098+8102+8106+8110+8114+8118+8122+8126+8130+8134+8138+8142+8146+8150+8154+8158+8162+8166+8170+8174+8178+8182+8186+8190+8194+8198+8202+8206+8210+8214+8218+8222+8226+8230+8234+8238+8242+8246+8250+8254+8258+8262+8266+8270+8274+8278+8282+8286+8290+8294+8298+8302+8306+8310+8314+8318+8322+8326+8330+8334+8338+8342+8346+8350+8354+8358+8362+8366+8370+8374+8378+8382+8386+8390+8394+8398+8402+8406+8410+8414+8418+8422+8426+8430+8434+8438+8442+8446+8450+8454+8458+8462+8466+8470+8474+8478+8482+8486+8490+8494+8498+8502+8506+8510+8514+8518+8522+8526+8530+8534+8538+8542+8546+8550+8554+8558+8562+8566+8570+8574+8578+8582+8586+8590+8594+8598+8602+8606+8610+8614+8618+8622+8626+8630+8634+8638+8642+8646+8650+8654+8658+8662+8666+8670+8674+8678+8682+8686+8690+8694+8698+8702+8706+8710+8714+8718+8722+8726+8730+8734+8738+8742+8746+8750+8754+8758+8762+8766+8770+8774+8778+8782+8786+8790+8794+8798+8802+8806+8810+8814+8818+8822+8826+8830+8834+8838+8842+8846+8850+8854+8858+8862+8866+8870+8874+8878+8882+8886+8890+8894+8898+8902+8906+8910+8914+8918+8922+8926+8930+8934+8938+8942+8946+8950+8954+8958+8962+8966+8970+8974+8978+8982+8986+8990+8994+8998+9002+9006+9010+9014+9018+9022+9026+9030+9034+9038+9042+9046+9050+9054+9058+9062+9066+9070+9074+9078+9082+9086+9090+9094+9098+9102+9106+9110+9114+9118+9122+9126+9130+9134+9138+9142+9146+9150+9154+9158+9162+9166+9170+9174+9178+9182+9186+9190+9194+9198+9202+9206+9210+9214+9218+9222+9226+9230+9234+9238+9242+9246+9250+9254+9258+9262+9266+9270+9274+9278+9282+9286+9290+9294+9298+9302+9306+9310+9314+9318+9322+9326+9330+9334+9338+9342+9346+9350+9354+9358+9362+9366+9370+9374+9378+9382+9386+9390+9394+9398+9402+9406+9410+9414+9418+9422+9426+9430+9434+9438+9442+9446+9450+9454+9458+9462+9466+9470+9474+9478+9482+9486+9490+9494+9498+9502+9506+9510+9514+9518+9522+9526+9530+9534+9538+9542+9546+9550+9554+9558+9562+9566+9570+9574+9578+9582+9586+9590+9594+9598+9602+9606+9610+9614+9618+9622+9626+9630+9634+9638+9642+9646+9650+9654+9658+9662+9666+9670+9674+9678+9682+9686+9690+9694+9698+9702+9706+9710+9714+9718+9722+9726+9730+9734+9738+9742+9746+9750+9754+9758+9762+9766+9770+9774+9778+9782+9786+9790+9794+9


```

D=16; J=1
NUMERATOR:
+1121+7922+40223+151724+477325+1309126+3179227+7095828+14718829+286483210+328281211+1528867212+1566654213+2545270714+3999324215+609545321
+690359717+13056351718+18424514719+25431928720+34387351721+44560041022+59369612723+75959365724+95584689725+118383340126+144403348727+1
73575948228+2057086729+240468763230+277386338731+315949844732+355128926733+394381597234+432692894735+469097175736+502627550237+5323430
91738+557385451739+577002474740+590597620241+597745644742+59822624743+592020018744+579325932745+560535078746+536226434747+507126269248+
47408678249+438832199250+39993503251+350758159252+32130462253+282801486754+245624127255+21022607256+177991787257+148380521758+121903
99875949864627260+78573486261+61560874762+47404566263+35843599264+25584986265+19317953766+13734402767+9238097268+6458894769+4255029770+
272047271+168267772+10034332773+574154274+313678775+162438276+79121777+35809278+14882779+5541780+1810281+491282+104783+13284+1285
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)

D=16; J=2
NUMERATOR:
+471+35622+14023+442524+106975+232876+455477+816828+1359979+21277210+31506211+44480212+66195713+78438214+98764215+120563216+143018217+1651
79218+186011719+20445120+21940471+230189722+23596023+23549472+231508723+221576726+207211727+139891728+166774229+146756730+124284731+
102320732+81689733+63071234+46915735+33472236+22793737+14714238+8931730+5052740+2621741+1225742+503743+173744+443745+8746
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)

D=16; J=3
NUMERATOR:
+972+6223+28324+95425+270376+557177+1442978+2899129+54280710+95604211+16000712+255586213+39268714+581591715+8333770716+1160136717+15708
07718+2072980719+2671551220+3356270221+415258722+5019504723+5951135724+692458525+791342226+885860727+980887428+1064765529+11370122
230+11494581731+12351438732+12503707733+12568905734+12410181735+1249762737+10502545738+9984325739+9075639240+811010041+712
278247+6143090243+5201421744+4319875245+3516592746+280277327+12135064748+1663606749+1235321750+8927302751+626603752+4255881753+279513754+
176283255+106418256+61051257+3052758+16600259+770760+3244761+1198762+367753+88764+12765+1766
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)

D=16; J=4
NUMERATOR:
+373+1274+3775+96276+19477+35628+59379+917710+1314211+1803212+2332213+2907714+3466215+3997216+4423217+4764718+4931719+4959720+4809721+453
422+4102223+3612724+3042225+2480226+1924227+1438228+1091229+670230+467231+230732+111733+52734+15235+5736
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)

D=16; J=5
NUMERATOR:
+175+226+1347+2828+6279+115210+209711+329712+520713+738214+1035215+1363216+1757217+2150718+2594719+2381720+3368721+3661722+3907723+440092
24+4055725+3938726+3763727+3465728+3128729+2707730+2313731+1877732+1498733+1132734+838735+576736+395737+241738+144239+76240+39741+13742+
7243+1744
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)

D=16; J=7
NUMERATOR:
+1728
DENOMINATOR:
(1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-30)(1-31)(1-32)(1-33)(1-34)(1-35)(1-36)(1-37)(1-38)(1-39)(1-40)(1-41)(1-42)(1-43)(1-44)(1-45)(1-46)(1-47)(1-48)(1-49)(1-50)(1-51)(1-52)(1-53)(1-54)(1-55)(1-56)(1-57)(1-58)(1-59)(1-60)(1-61)(1-62)(1-63)(1-64)(1-65)(1-66)(1-67)(1-68)(1-69)(1-70)(1-71)(1-72)(1-73)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-80)(1-81)(1-82)(1-83)(1-84)(1-85)

*****THE POINCARÉ SERIES FOR D=15*****
NUMERATOR:
+170+173+124+275+876+1177+3028+5979+112210+204711+389712+655713+1146714+1896715+3067716+4823717+7477718+11167719+16506720+23764721+33593

```

722+46576723+63598724+85049725+112273726+145668727+186298728+234773729+292016730+357883731+433493732+518058733+611720734+713572735+82299
 7236+937742737+1057289738+1178403739+1299472740+1417560741+1530503742+1634576743+1728546744+1808808745+1873856746+1921520747+1951002748+
 1960390749+1951002750+1921520751+1873856752+1808803753+1728646754+1634576755+1530508756+1417560757+1299472758+1178403759+1057269260+9377
 42761+82299762+713572763+611720764+518058765+433493766+357883767+292016768+234773769+186298770+145668771+112273772+85049773+63598774+46
 576275+33593776+23764777+16505778+11167279+7477280+4823781+3067282+1896283+1146284+655285+389286+204787+112268+59289+30290+11191+8792+27
 93+1294+1295+1298
 DENOMINATOR:
 (1-72)(1-74)(1-75)(1-76)(1-77)(1-78)(1-79)(1-710)(1-711)(1-712)(1-713)(1-715)

